**TITLE**: The Steady Linear Response of a Spherical Atmosphere to Thermal and Orographic Forcing

**AUTHOR**: Brian J. Hoskins
David J. Karoly

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**Reasons for Review:**

- What are the patterns of waves induced by large-scale forcing?
- How does a spherical atmosphere respond to thermal and orographic forcing?
- Linear, steady-state assumptions provide a good guide to predicting the large-scale pattern of perturbations induced by large-scale forcing
- Subtropical forcing has important implications for the middle latitudes

**Abbreviations/Symbols:**

- H High Pressure
- $K_S$ Local Stationary Wavenumber
- L Low Pressure
- NH Northern Hemisphere
- NL # of Model Levels
- PV Potential Vorticity

**Model Details:**

- Hemispheric baroclinic model
    - Linearized and steady state
    - 5 vertical layers
        - Not enough to capture longest waves
- Basic numerical model
    - Primitive equation; σ-coordinate
        - Hoskins and Simmons (1975)
    - Spectral-transform technique in horizontal

- $2^{nd}$ order finite differences in the vertical
- Good for study of baroclinic instability
- Zonal wavenumber (*m*) is independent
- Vorticity ($\xi$)

$$\xi = \left[\xi_{m+1}^{j,m} P_{m+1}^{m}(\mu) + \xi_{m+3}^{j,m} P_{m+3}^{m}(\mu) + \cdots + \xi_{m+J}^{j,m} P_{m+J}^{m}(\mu)\right] e^{im\lambda}$$

- Divergence (*D*)

$$D = \left[D_{m}^{j,m} P_{m}^{m}(\mu) + D_{m+2}^{j,m} P_{m+2}^{m}(\mu) + \cdots + D_{m+J-1}^{j,m} P_{m+J-1}^{m}(\mu)\right] e^{im\lambda}$$

- Zonal Wavenumber (*m*) may be described by the vector:

$$\boldsymbol{X} = (\boldsymbol{\xi}, i\boldsymbol{D}, \boldsymbol{T}, \ln \boldsymbol{p}_*)$$

- Linearized primitive equations with no source or sink terms:

$$\dot{\boldsymbol{X}} = iA\boldsymbol{X}$$

  - *A* is a square matrix of side (3 x NL + 1) x (J + 1)/2
  - $r_n$ factors are included to make more elements of A of order 1.

- Steady problem
  - $0 = iA\boldsymbol{X} + D\boldsymbol{X} + \boldsymbol{F}$
  - $\boldsymbol{X} = i(A - iD)^{-1}\boldsymbol{F}$ (steady response to forcing **F**)
- Dissipation on $\xi$
  - $\lambda_U(\sigma)\xi - K\nabla_h^4\xi$
- Dissipation on D
  - $\lambda_T(\sigma)T' - K\nabla_h^4 T'$
  - λ → effects of boundary layer
  - K = 2.338e16 $m^4 s^{-1}$.
- Temperature perturbation
  - $T' = T - \ln p_* \frac{dT(\sigma)}{d \ln\sigma}$
  - Dissipation matrix D is not diagonal
- Vertical Levels
  - $p/p_*$ = 0.1, 0.3, 0.5, 0.7, 0.9
- Spectral truncation (J) = 25
  - Each zonal wavenumber and level of each variable is described by 13 complex coefficients
- 26 zonal wavenumbers can be resolved
  - Variance in high wavenumbers is small, so they are often neglected
- Zonal flow
  - Symmetry about equator

  - Oort and Rasmusson (1971)

**Notes:**

- First accounts of steady linear response of atmosphere
  - Charney and Eliassen (1949)
  - Smagorinsky (1953)
- Study of orographic forcings
  - Grose and Hoskins (1979)
    - Barotropic model to study Rossby response in zonal flows induced by simple mountains
- Study of thermal forcings
  - Difficult in midlatitudes due to baroclinicity
  - In tropics, warmer SSTs may induce extra convective heating
- One typical case
  - Zonal flow → NH 300 mb winter flow
  - Elliptical negative vorticity source
    - equator and 30°N; 30°W and 30°E
  - Total vorticity source = 0
    - Uniform positive vorticity source is placed elsewhere at same latitudes
  - Solutions are linearly proportional to the magnitude of forcing
  - Max upper level divergence = 4e-6 $s^{-1}$.
    - Extra 10 mm of rain per day
  - After 10 days
    - Solutions don't change much
    - Propagation to the NORTH and EAST from source
      - Alternating H and L centers
      - Begins with H just to the west of the forcing center
  - Agreement with Bjerknes (1966)
    - Strengthening of the subtropical jet
    - Trough to the north
- Patterns reveal an equivalent barotropic structure
  - Largest amplitudes in upper troposphere

- Thermal forcing
  - Thermal sources differ by latitude and vertical distribution
  - β-plane vorticity equation
    - $\bar{u}\xi'_x + \beta v' = f w'_z$
  - Potential temperature equation
    - $\bar{u}\theta'_x + v'\bar{\theta}_y + w'\bar{\theta}_z = \frac{\theta_0}{g} Q$
    - $f\bar{u}v'_z + f\bar{u}_z v' + w'N^2 = Q$
  - Tropics

        - γ (Held 1978) is small
        - Near-surface heating results in extremely large *v′*
        - Heat source away from surface balanced by upward motion
        - **Surface trough must be to the west of thermal source**
    - Midlatitudes
        - γ is large
        - Heating at any level is balanced by hor. temperature advection
        - If $H_Q >> H_u$
            - Deep case
            - Creation of vorticity (via β) must be balanced by vortex shrinking
        - If $H_Q \sim H_u$
            - Shallow case
            - Low-level heating balanced by zonal temperature advection
        - **Surface trough must be to the east of the thermal source**
    - Subtropical forcing
        - *15°N is a subtropical forcing??*
        - Heat source
            - Centered @ 500mb, 15°N
            - Elliptical (long way in x-direction)
            - Indicative of deep convective profile
        - Low-level damping to represent boundary layer
        - Max vertically-averaged heating rate = 2.5 K $day^{-1}$.
        - To the west
            - H above the max forcing level
            - L below the max forcing level
        - 300mb Vorticity
            - Very similar to barotropic model response (Fig. 1)
            - Wave train moves north and east
                - Split right near north pole
                    - Longer wavenumbers continue over the top of the globe
                    - Smaller wavenumbers turn back toward the equator
            - Height field emphasizes the "polar" wave train
    - Mid-latitude forcing
        - Heat source
            - Centered at 45°N
            - Circular, but same area as ellipse in subtropical section
            - Rather large for midlatitudes
        - Surface L is located 18° to the east of the source
        - Upper-level H is located 60° to the east

        - Wave train initializes 30° downstream (to the east)
            - Split still occurs near north pole
        - Height field distribution is very similar to subtropical forcing
            - If elliptical heat source is used, results are nearly identical
        - Easier to force a height field anomaly from the subtropics than from the mid and high latitudes
        - 900mb T' is negative in the region of heating
            - Based on thermal wind relationship
        - Shallow heating
            - If heating is centered near the surface (instead of 500mb)
            - Surface L is strengthened
            - T' is now positive near the heat source
            - **Positive vorticity decreases with height, which leads to positive T'**
            - Shallow heating is more common in midlatitudes
        - Deep heating
            - **Positive vorticity increases with height, which leads to negative T'**
    - Other cases
        - Heat Source on equator
            - Response away from equator is small
        - Between 20°N and 30°N, the balance of heating by vertical motion is replaced by the balance of heating by horizontal advection
        - Doubling heat radius at 45°N doubles the height anomalies
        - Shallow heat source has a much larger low-level response if placed at 60°N rather than 45°N.
            - Smaller T gradient, so stronger meridional winds are required for same balance at lower latitudes
        - Damping experiments
            - Fluxes of heat and PV dominate
            - Damping temperature and/or velocity does not have a major impact on the solutions
        - 0.7*U
            - Poleward and eastward train with slightly shorter wavelength
            - As U is increased (increase 0.7 factor), more structure appears
        - Adding constant angular velocity (3 m $s^{-1}$) at the equator does not change the solution by much
- Orographic Forcing
    - Barotropic Case
        - Columns are squashed on the upslope

            - Generates anticyclonic vorticity
        - Columns are stretched on the downslope
            - Generates cyclonic vorticity
        - Long wavelengths
            - β is dominant
            - Cyclone over the ridge
        - Short and medium wavelengths
            - Zonal vorticity advection dominates
            - Anticyclone over the ridge
        - $k_S$ is taken from mid- to upper- troposphere
            - $k_S$ = 7000 km
            - Shortwave case is most relevant since most waves are shorter than 7000 km
    - Baroclinic Case
        - Anticyclone always sits (approx.) over the mountain
            - Upslope cooling compensated by southerly flow (warm)
            - Downslope warming compensated by northerly flow (cold)
        - Long wavelengths
            - Waves are:
                - Vertically-propagating
                - Slope westward with height
                - Warmest air ahead of trough
            - Anticyclone is slightly displaced to the west of mountain
        - Shorter wavelengths
            - Waves are trapped with H over mountain and troughs to either side
        - $k_S$ is taken from lower troposphere
            - $k_S$ = 3000 km
            - Longwave case is most relevant since most waves are longer than 3000 km
    - Barotropic theory with damping and baroclinic theory tend to give the same pressure phase relationship over a mountain
    - Circular mountain at 30°N
        - Upper-level damping for loss of wave activity to stratosphere
        - Mountain diameter = 45° of latitude
        - Mountain height = 2 km
        - Mid-latitudes ( > 30°N)
            - Column wind is westerly
            - Surface H is 3° west of the mountain
            - Surface L is 33° east of the mountain
            - Westward tilts with height
            - Max amplitudes in upper troposphere
        - Low latitudes ( < 30°N)

                - Low-level wind is easterly, mid- to upper-level wind is westerly
                - Surface H is 18° to the east
                - Surface L is 9° to the west
                - Pressure pattern aloft is consistent with mid-latitude response
            - 2 wave trains
                - Longwave polar wavetrain
                - Shortwave subtropical wavetrain
        - Other mountain cases
            - Adding 3 m $s^{-1}$ to global zonal wind
                - Shrinks region of equatorial easterlies
                - Wavetrains have slightly greater wavelength
            - Reduce horizontal mountain dimensions by $\sqrt{2}$
                - Similar pattern
                - Polar wavetrain amplitude reduced by factor of 4
            - Move mountain to 60°N
                - Similar polar wavetrain
                - Equatorward wavetrain of long wavelength
            - Elongated mountain (area conserved)
                - Amplitude of long wavelengths is reduced
                - Increased amplitude near surface westerlies
        - Earth orography
            - Himalayas, followed by Rockies, are responsible for observed height field perturbations
            - Shorter wavelength differences
                - Particularly in Pacific sector
                - Maybe linearized theory is not valid
                    - Ambient flow stronger than zonal average
                - Aleutian and Atlantic L are likely thermally forced
                    - Orographic uplift may be directly associated with Siberian H and west U.S. H

- Important Points
    - **Away from the source region the wavetrains produced in the upper troposphere of the baroclinic model and in the barotropic models are qualitatively and even quantitatively very similar.**
        - Wavetrains in baroclinic model have equivalent barotropic structure.
        - Amplitudes are largest in upper troposphere
    - **Long wavelengths propagate strongly poleward as well as eastward**
        - Wavetrain is like a great circle

- **Shorter wavelengths appear to be trapped equatorward of the poleward flank of the midlatitude jet**
  - Splitting of the wavetrains
  - Possible "blocking" region downstream where long, poleward wavetrain and short, subtropical wavetrain are out of phase
- **Easiest way to reproduce amplitude in mid and high latitudes is by subtropical forcing**
- **The low-level temperature field of midlatitude thermal forcing is crucially dependent on the vertical distribution of the source.**
  - Explained via thermal wind relation

- Rossby Wave Rays
  - Barotropic Rossby waves in a slowly varying medium
    - Horizontal streamfunction perturbation

$$\left(\frac{\partial}{\partial t} + \bar{u}_M \frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) + \beta_M \frac{\partial \psi}{\partial x} = 0 \qquad \textbf{(5.9)}$$

$$\beta_M = \frac{2\Omega}{a} cos^2\varphi - \frac{d}{dy}\frac{1}{cos^2\varphi}\frac{d}{dy}(cos^2\varphi \bar{u}_M)$$

    - Dispersion relation

$$\omega = \bar{u}_M k - \frac{\beta_M k}{k^2+l^2} \quad \text{(from 5.9)}$$

    - Group velocities

$$\boldsymbol{c_g} = (u_g, v_g)$$

$$u_g = \frac{\partial \omega}{\partial k} = \frac{\omega}{k} + \frac{2\beta_M k^2}{(k^2+l^2)^2}$$

$$v_g = \frac{\partial \omega}{\partial l} = \frac{2\beta_M kl}{(k^2+l^2)^2}$$

      - **Ray** is defined to be everywhere in the direction of the local $\boldsymbol{c_g}$.
        - Energy propagated along a ray at the group velocity
        - *k* and $\omega$ are constant along a ray; *l* varies
    - For $\omega = 0$
      - Slope = *l*/*k*
      - $\boldsymbol{c_g} = 2\frac{k}{K_S}\bar{u}_M$
        - Energy propagates at double the background basic flow
    - Use a solution of the form $\psi = P(y)e^{i(kx-\omega t)}$ and plug into (5.9)
      - See derivations
    - The latitude where $K_S = k$ provides a turning point for the ray
    - If $\bar{u} = 0$
      - That latitude is where rays become more meridional
      - Group velocity tends to 0
  - Constant angular velocity flow

        - See derivations for **(5.24)**
        - The fastest energy propagation time around the sphere is ~15.5 days
        - At any turning point, the wavelength is 5000km
        - Resonance affects the amplitude of the downstream wavetrain but otherwise has a small role.
        - In applying wave theories, the zonal periodicity of the sphere should be dropped.
    - More realistic flows
        - Longer wavelengths propagate poleward
            - Wavenumbers 1-3
        - Shorter wavelengths are trapped in the northern flank of the midlatitude jet and do not propagate poleward
            - Wavenumbers 4+
        - Move the source poleward (to 45°N)
            - Only wavenumbers 1-3
        - Details of the basic flow are unimportant to the solutions given
        - Using 500 mb zonal flow instead of 300 mb zonal flow allows wavenumber 4 train to be more poleward
            - 5 and 6 are not as trapped either
        - If the flow is increased (1.2*u), wavenumber 3 train has a chance to go all the way around the sphere
- Dissipation
    - When small, resonance is important and makes the resulting structures sensitive
    - When ideal, no sensitivity for the structure of the responses
    - When larger, same patterns but reduced amplitude

## Derivations

Derive **(5.24)**:

$$\beta_M = \frac{2\Omega}{a}cos^2\varphi - \frac{d}{dy}\frac{1}{cos^2\varphi}\frac{d}{dy}(cos^2\varphi\bar{u}_M) \qquad (5.10)$$

Constant angular velocity flow:

$$\bar{u}_M = a\widetilde{\omega}$$

$$\beta_M = \frac{2\Omega}{a}cos^2\varphi - \frac{d}{dy}\frac{1}{cos^2\varphi}\frac{d}{dy}(cos^2\varphi a\widetilde{\omega})$$

Replace $\frac{d}{dy}$ with $\frac{d}{d\varphi}$:

$$\frac{d}{dy} = \frac{cos\varphi}{a}\frac{d}{d\varphi}$$

$$\beta_M = \frac{2\Omega}{a}cos^2\varphi - \frac{cos\varphi}{a}\frac{d}{d\varphi}\frac{1}{cos^2\varphi}\frac{cos\varphi}{a}\frac{d}{d\varphi}(cos^2\varphi a\widetilde{\omega}) \qquad \text{...simplify...}$$

$$\beta_M = \frac{2\Omega}{a}cos^2\varphi - \frac{cos\varphi}{a}\frac{d}{d\varphi}\frac{1}{acos\varphi}\frac{d}{d\varphi}(cos^2\varphi a\widetilde{\omega})$$

Take first derivative:

$$\beta_M = \frac{2\Omega}{a}cos^2\varphi - \frac{cos\varphi}{a}\frac{d}{d\varphi}\frac{1}{acos\varphi}(-2cos\varphi sin\varphi a\widetilde{\omega}) \qquad \text{...simplify...}$$

$$\beta_M = \frac{2\Omega}{a}cos^2\varphi - \frac{cos\varphi}{a}\frac{d}{d\varphi}(-2sin\varphi\widetilde{\omega})$$

$$\beta_M = \frac{2\Omega}{a}cos^2\varphi - (-2\widetilde{\omega})\frac{cos\varphi}{a}\frac{d}{d\varphi}(sin\varphi)$$

Take second derivative:

$$\beta_M = \frac{2\Omega}{a}cos^2\varphi + 2\widetilde{\omega}\frac{cos\varphi}{a}cos\varphi$$

$$\boldsymbol{\beta_M = \frac{2cos^2\varphi}{a}(\Omega + \widetilde{\omega})} \qquad \textbf{(5.24)}$$

Derive **(5.11)**

$$\left(\frac{\partial}{\partial t} + \bar{u}_M \frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) + \beta_M \frac{\partial \psi}{\partial x} = 0 \qquad \textbf{(5.9)}$$

Solve for $\psi$:

$$\psi = e^{i(kx+ly-\omega t)}$$

Plug and chug:

$$\left(\frac{\partial}{\partial t} + \bar{u}_M \frac{\partial}{\partial x}\right)(i^2k^2 + i^2l^2) + i\beta_M k = 0$$

$$i\omega k^2 - i\bar{u}_M k^3 + i\omega l^2 - i\bar{u}_M k l^2 + i\beta_M k = 0$$ …drop the i's…

$$\omega(k^2 + l^2) = \bar{u}_M k(k^2 + l^2) - \beta_M k$$

$$\omega = \bar{u}_M k - \frac{\beta_M k}{(k^2+l^2)} \qquad \textbf{(5.11)}$$

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Derive **(5.12)**

$$u_g = \frac{\partial \omega}{\partial k} = \frac{\partial}{\partial k}\left(\bar{u}_M k - \frac{\beta_M k}{(k^2 + l^2)}\right)$$

$$\frac{\partial}{\partial k}\left(\frac{\beta_M k}{(k^2+l^2)}\right) = \frac{\beta_M}{(k^2+l^2)} - \frac{2\beta_M k^2}{(k^2+l^2)^2}$$

$$u_g = \bar{u}_M + \frac{\beta_M}{(k^2 + l^2)} - \frac{2\beta_M k^2}{(k^2 + l^2)^2}$$

$$u_g = \frac{\omega}{k} - \frac{2\beta_M k^2}{(k^2+l^2)^2} \qquad \textbf{(5.12)}$$

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Derive **(5.13)**

$$v_g = \frac{\partial \omega}{\partial l} = \frac{\partial}{\partial l}\left(\frac{\beta_M k}{(k^2 + l^2)}\right)$$

$$v_g = -\frac{2\beta_M kl}{(k^2+l^2)^2} \qquad \textbf{(5.13)}$$

Derive **(5.19):**

$$\left(\frac{\partial}{\partial t}+\bar{u}_M\frac{\partial}{\partial x}\right)\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+\beta_M\frac{\partial\psi}{\partial x}=0 \qquad \textbf{(5.9)}$$

Solve for $\psi$:

$$\psi = P(y)e^{i(kx-\omega t)}$$

$$\left(\frac{\partial}{\partial t}+\bar{u}_M\frac{\partial}{\partial x}\right)\left(i^2k^2P+\frac{d^2P}{dy^2}\right)+i\beta_M kP=0$$

$$i\omega k^2P-ik^3\bar{u}_MP-i\omega\frac{d^2P}{dy^2}+i\bar{u}_Mk\frac{d^2P}{dy^2}+i\beta_MkP=0$$

$$\frac{d^2P}{dy^2}(\bar{u}_Mk-\omega)+P(-\bar{u}_Mk^3+\omega k^2+\beta_Mk)=0$$

$$\frac{d^2P}{dy^2}(\bar{u}_Mk-\omega)+P(-k^2(\bar{u}_Mk-\omega)+\beta_Mk)=0$$

$$\frac{1}{(\bar{u}_Mk-\omega)}\left[\frac{d^2P}{dy^2}(\bar{u}_Mk-\omega)+P(-k^2(\bar{u}_Mk-\omega)+\beta_Mk)=0\right]$$

$$\frac{d^2P}{dy^2}+P\left(\frac{\beta_Mk}{(\bar{u}_Mk-\omega)}-k^2\right)=0$$ …Multiply by $\frac{\frac{1}{\bar{u}_Mk}}{\frac{1}{\bar{u}_Mk}}$…

$$\frac{d^2P}{dy^2}+P\left(\frac{\frac{\beta_Mk}{\bar{u}_Mk}}{\left(1-\frac{\omega}{\bar{u}_Mk}\right)}-k^2\right)=0$$

Define $K_S$:

$$K_S=\left(\frac{\beta_M}{\bar{u}_M}\right)^{\frac{1}{2}}$$

$$l^2(y)=\frac{{K_S}^2}{\left(1-\frac{\omega}{\bar{u}_Mk}\right)}-k^2$$

$$\frac{d^2P}{dy^2}+l^2(y)P=0 \qquad \textbf{(5.19)}$$